Arbitrarily Laminated, Anisotropic Cylindrical Shell Under Internal Pressure

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An arbitrarily laminated, anisotropic cylindrical shell of finite length, under uniform internal pressure, is analyzed using Love-Timoshenko's kinematic relations and under the framework of classical lamination theory. The previously obtained solutions for asymmetrically laminated orthotropic (cross-ply) as well as unbalanced-symmetric and balanced-unsymmetric (angle-ply) cylindrical shells under the same loading conditions have been shown to be special cases of the present closed-form solution. Numerical results have been presented for a two-layer cylindrical shell and compared with those obtained using finite element solutions based on the layerwise constant shear-angle theory. These are expected to serve as benchmark solutions for future comparisons and to facilitate the use of unsymmetric lamination in design.

Nomenclature

A_{ij},B_{ij},D_{ij}	= extensional, coupling, and bending rigidities, respectively, for $i,j = 1, 2, 6$
L	= length of the circular cylindrical shell
$M_x,M_\theta,M_{x\theta},M_{\theta x}$	= bending and twisting moments per unit length
$N_x,N_\theta,N_{x\theta},N_{\theta x}$	= in-plane normal and shearing stress resultants per unit length
p	= uniform internal pressure
Q_x,Q_θ	= transverse shear stress resultants per unit length
R	= radius of the reference (middle) surface of the cylindrical shell
t	= thickness of the laminated shell
u,v	= surface-parallel displacement components in longitudinal and circumferential directions, respectively
u_0,v_0	= reference (middle) surface stretching displacements
w	= transverse or radial displacement
x,θ,z	= right-hand cylindrical shell coordinates
$\epsilon_1,\epsilon_2,\gamma_{12}$	= reference (middle) surface normal and shearing strains
$\bar{\theta}_i$	= fiber orientation angle of the ith layer
χ_1,χ_2,χ_{12}	= changes of curvature and twist
$\sigma_x,\sigma_\theta,\tau_{x\theta}$	= surface-parallel normal and shearing stress components

Introduction

THE increasing use of fiber-reinforced laminated composite shells in aerospace as well as ground applications and the various shell theories employed for analyzing them are now fairly well documented.[1-4] The analysis of these shell structures is fraught with such complications as introduced by the anisotropy of individual laminae and the asymmetry of, in general, lamination that result in various coupling effects. The first study that incorporated the bending/stretching coupling due to unsymmetric lamination was by Ambartsumyan,[5] who considered orthotropic layers. Dong et al.[6] applied a theory of a thin laminated anisotropic shell, which may be considered to be an extension of work of Reissner and Stavsky[7] on laminated anisotropic plate, to Donnell's shallow shell theory. This theory was utilized by Reuter[8] in obtaining closed-form solutions for the unbalanced-symmetric and balanced-unsymmetric angle-ply cylindrical shells of finite length under the separate influences of a uniform internal pressure and a uniform temperature change. Balaraman et al.[9] obtained closed-form solutions for asymmetrically laminated cylindrical shells of finite length, consisting of orthotropic as well as anisotropic layers and under uniform internal pressure. However, numerical results were presented for asymmetrically laminated orthotropic layers only.[9,10] Reference 9 also presented a series solution for an asymmetrically laminated cylindrical shell consisting of orthotropic layers and subjected to nonaxisymmetric loadings. Extensive numerical results were not presented, however. Recently, Bert and Reddy[11] have presented exact solutions for bending under sinusoidal transverse loading of two-layer thin cylindrical shells. Reddy[12] has used the constant-shear angle theory (CST) to present series solutions for cross-ply shallow shells of cylindrical as well as doubly curved geometries utilizing Sanders' kinematic relations.

A brief review of the published literature suggests that an analytical solution is not available for an arbitrarily laminated anisotropic cylindrical shell, even under such a simplifying assumption as the classical lamination theory (CLT), based on Love's first approximation theory[13] (also known as the Love-Kirchhoff theory). As a result, designers of laminated composite shell structures shy away from the employment of unsymmetric lamination of anisotropic layers, which may offer important additional advantages in practical design over the more traditional cross- and angle-ply constructions. The present paper will, therefore, try to fill this gap. CLT and Love-Timoshenko's kinematic relations[14,15] will be the basis of the present investigation. The numerical results, obtained using the present closed-form solution will be compared with those, computed using triangular finite elements based on the layer constant shear-wise-angle theory (LCST).[4,14] The purpose of this comparison is to evaluate the applicability of CLT in case of asymmetrically laminated, anisotropic cylindrical shells.

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Derivation of Closed-Form Solution

The equations of equilibrium for a circular cylindrical shell (Fig. 1) subjected to uniform internal pressure p are given by[10,15,16]

$$N_{x,x}+\frac{1}{R}N_{\theta x,\theta}=0 \tag{1a}$$

$$N_{x\theta,x}+\frac{1}{R}N_{\theta,\theta}+\frac{Q_\theta}{R}=0 \tag{1b}$$

$$\frac{N_\theta}{R}-Q_{x,x}-\frac{1}{R}Q_{\theta,\theta}=p \tag{1c}$$

$$M_{x,x}+\frac{1}{R}M_{\theta x,\theta}=Q_x \tag{1d}$$

$$M_{x\theta,x}+\frac{1}{R}M_{\theta,\theta}=Q_\theta \tag{1e}$$

$$N_{x\theta}-N_{\theta x}-\frac{M_{\theta x}}{R}=0 \tag{1f}$$

where the strain-displacement relations are[10,15,17]

$$\epsilon_1=u_{0,x};\quad \epsilon_2=\frac{1}{R}(V_{0,\theta}+w);\quad \gamma_{12}=v_{0,x}+\frac{1}{R}u_{0,\theta} \tag{2a}$$

$$\chi_1=-w_{,xx};\quad \chi_2=\frac{1}{R^2}(-w_{,\theta\theta}+v_{0,\theta})$$

$$\chi_{12}=\frac{2}{R}(-w_{,x\theta}+v_{0,x}) \tag{2b}$$

The stress resultants and stress couples can be expressed in terms of the reference surface strains and curvatures as follows[10,17]:

$$N_x=A_{11}\epsilon_1+A_{12}\epsilon_2+A_{16}\gamma_{12}+B_{11}\chi_1+B_{12}\chi_2+B_{16}\chi_{12} \tag{3a}$$

$$N_\theta=A_{12}\epsilon_1+A_{22}\epsilon_2+A_{26}\gamma_{12}+B_{12}\chi_1+B_{22}\chi_2+B_{26}\chi_{12} \tag{3b}$$

$$N_{x\theta}=A_{16}\epsilon_1+A_{26}\epsilon_2+A_{66}\gamma_{12}+B_{16}\chi_1+B_{26}\chi_2+B_{66}\chi_{12}$$
$$+(1/R)(B_{16}\epsilon_1+B_{26}\epsilon_2+B_{66}\gamma_{12}+D_{16}\chi_1+D_{26}\chi_2+D_{66}\chi_{12}) \tag{3c}$$

$$N_{\theta x}=A_{16}\epsilon_1+A_{26}\epsilon_2+A_{66}\gamma_{12}+B_{16}\chi_1+B_{26}\chi_2+B_{66}\chi_{12} \tag{3d}$$

$$M_x=B_{11}\epsilon_1+B_{12}\epsilon_2+B_{16}\gamma_{12}+D_{11}\chi_1+D_{12}\chi_2+D_{16}\chi_{12} \tag{3e}$$

$$M_\theta=B_{12}\epsilon_1+B_{22}\epsilon_2+B_{26}\gamma_{12}+D_{12}\chi_1+D_{22}\chi_2+D_{26}\chi_{12} \tag{3f}$$

$$M_{x\theta}=M_{\theta x}=B_{16}\epsilon_1+B_{26}\epsilon_2+B_{66}\gamma_{12}+D_{16}\chi_1+D_{26}\chi_2+D_{66}\chi_{12} \tag{3g}$$

The expression for $N_{x\theta}$ as given by Eq. (3c) has been chosen such that the sixth equation of equilibrium is satisfied.

Eliminating Q_x and Q_θ from Eqs. (1), then substituting Eqs. (2) and (3) into them and noting that, because of axisymmetry $\partial/\partial\theta(\cdots)=0$, but $v_0\neq 0$, will yield

$$A_{11}u_{0,xx}+\frac{A_{12}}{R}w_{,x}+A_{16}v_{0,xx}-B_{11}w_{,xxx}+\frac{2B_{16}}{R}v_{0,xx}=0 \tag{4a}$$

$$A_{16}u_{0,xx}+\frac{A_{26}}{R}w_{,x}+A_{66}v_{0,xx}-B_{16}w_{,xxx}+\frac{2B_{16}}{R}u_{0,xx}$$
$$+\frac{2B_{26}}{R^2}w_{,x}+\frac{4B_{66}}{R}v_{0,xx}-\frac{2D_{16}}{R}w_{,xxxx}+\frac{4}{R^2}D_{66}v_{0,xx}=0 \tag{4b}$$

$$\frac{A_{12}}{R}u_{0,x}+\frac{A_{22}}{R^2}w+\frac{A_{26}}{R}v_{0,x}-B_{11}u_{0,xxx}-\frac{2B_{12}}{R}w_{,xx}$$
$$-B_{16}v_{0,xxx}+\frac{2B_{26}}{R^2}v_{0,x}+D_{11}w_{,xxx}-\frac{2D_{16}}{R}v_{0,xxx}=p \tag{4c}$$

Equations (4a) and (4b) are then differentiated to obtain the expressions for $u_{0,xxx}$ and $v_{0,xxx}$ that are then substituted into Eq. (4c). These equations are also integrated to obtain

$$A_{11}u_{0,x}-\frac{A_{12}}{R}w+\left(A_{16}-\frac{2B_{16}}{R}\right)v_{0,x}-B_{11}w_{,xx}=A_1 \tag{5a}$$

$$\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)v_{0,x}=-\left(A_{16}+\frac{2B_{16}}{R}\right)u_{0,x}$$
$$-\frac{1}{R}\left(A_{26}+\frac{2B_{26}}{R}\right)w+\left(B_{16}+\frac{2D_{16}}{R}\right)w_{,xx}+A_2 \tag{5b}$$

where the integration constants A_1 and A_2 are equal to N_x and $N_{x\theta}+(M_{x\theta}/R)$, respectively. Eliminating $u_{0,x}$ and $v_{0,x}$ from Eqs. (5a), (5b), and (4) will finally yield

$$aw_{,xxxx}+bw_{,xx}+cw=p' \tag{6}$$

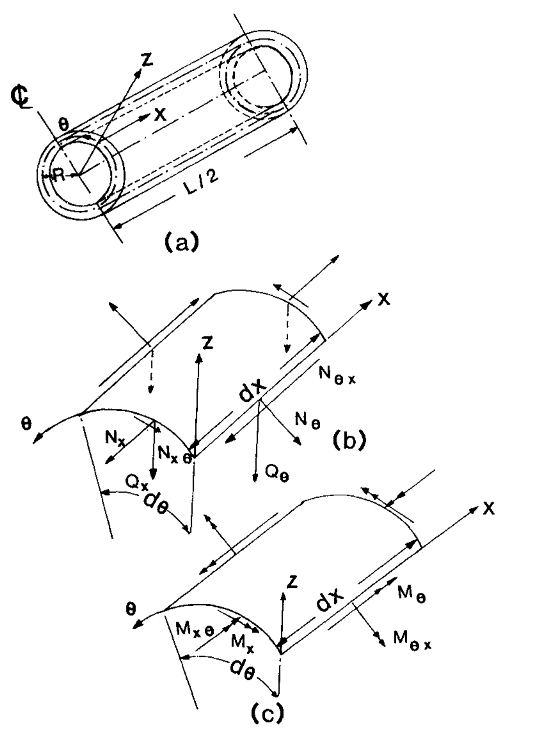

Fig. 1 A cylindrical shell: a) geometry, b) stress resultants, and c) stress couples.

where a, b, c, and p' are given by Eqs. (A1) in Appendix A. For orthotropic layers, they reduce to Eqs. (A2), which check with those obtained by Chaudhuri.[10] It is also easy to see that for unbalanced-symmetric and balanced-unsymmetric angle-ply shells, $b=0$ and Eqs. (6) and (A1) reduce to those obtained by Reuter.[8] The solution to Eq. (6) is of the form, obtained earlier by Paul[18] and Chaudhuri,[10] for asymmetrically laminated shells consisting of isotropic and orthotropic layers, respectively,

$$w=B_1\sinh\beta x\sin\alpha x+B_2\sinh\beta x\cos\alpha x+B_3\cosh\beta x\sin\alpha x + B_4\cosh\beta x\cos\alpha x+(p'/c) \tag{7}$$

where

$$\alpha=\left[\frac{1}{2}\left(\frac{c}{a}\right)^{1/2}+\frac{b}{4a}\right]^{1/2}, \quad \beta=\left[\frac{1}{2}\left(\frac{c}{a}\right)^{1/2}-\frac{b}{4a}\right]^{1/2} \tag{8}$$

In obtaining the solution given by Eq. (7), $b^2<4ac$ has been assumed, which is valid for most laminated shells used in practice. The constant b represents essentially a second-order beam/column effect[18] caused by the asymmetry of lamination.

The midsurface displacements u_0 and v_0 can now be obtained by substituting Eq. (7) into Eqs. (5) and then integrating with respect to x,

$$\begin{aligned} u_0 = {} & (A_1 a'-A_2 b')x+[B_2\alpha(c'-d') \\ & +B_3\beta(c'+d')]\sin\alpha x\sinh\beta x \\ & +[B_1\alpha(-c'+d')+B_4\beta(c'+d')]\cos\alpha x\sinh Bx \\ & +[B_1\beta(c'+d')+B_4\alpha(c'-d')]\sin\alpha x\cosh\beta x \\ & +[B_2\beta(c'+d')+B_3\alpha(-c'+d')]\cos\alpha\cosh\beta x+A_3 \end{aligned} \tag{9}$$

$$\begin{aligned} v_0 = {} & [-A_1 e'+A_2 f']x+[B_2\alpha(g'-h') \\ & +B_3\beta(g'+h')]\sin\alpha x\sinh\beta x \\ & +[B_1\alpha(-g'+h')+B_4\beta(g'+h')]\cos\alpha x\cosh\beta x \\ & +[B_1\beta(g'+h')+B_4\alpha(g'-h')]\sin\alpha x\cosh\beta x \\ & +[B_2\beta(g'+h')+B_3\alpha(-g'+h')]\cos\alpha x\cosh\beta x+A_4 \end{aligned} \tag{10}$$

where A_3 and A_4 are integration constants and a', b', c', d', e', f', g', and h' are as given by Eqs. (A3) and (A4).

The eight integration constants A_m and B_m, $m=1,\ldots,4$, can be obtained by prescribing four boundary conditions at each of the two ends $x=\pm L/2$ of the cylindrical shell. These can be described as follows:

Simply supported edge,

$$\text{SS1:}\quad w=M_x=N_x=N_{x\theta}=0 \tag{11a}$$

$$\text{SS2:}\quad w=M_x=u_0=N_{x\theta}=0 \tag{11b}$$

$$\text{SS3:}\quad w=M_x=N_x=v_0=0 \tag{11c}$$

$$\text{SS4:}\quad w=M_x=u_0=v_0=0 \tag{11d}$$

Clamped edge,

$$\text{CE1:}\quad w=w_{,x}=N_x=N_{x\theta}=0 \tag{12a}$$

$$\text{CE2:}\quad w=w_{,x}=u_0=N_{x\theta}=0 \tag{12b}$$

$$\text{CE3:}\quad w=w_{,x}=N_x=v_0=0 \tag{12c}$$

$$\text{CE4:}\quad w=w_{,x}=u_0=v_0=0 \tag{12d}$$

Free edge,

$$M_x=Q_x+(1/R)M_{x\theta,\theta}=N_x=N_{x\theta}=0 \tag{13}$$

For example, application of the boundary conditions SS1 at ends $x=\pm L/2$ will determine the integration constants as

$$A_1=A_2=A_3=A_4=0 \tag{14a}$$

because u_0 and v_0 are skew-symmetric functions of x,

$$p'=p \tag{14b}$$

$$B_1=-\frac{pd_1}{c(a_1d_1-b_1c_1)} \tag{14c}$$

$$B_4=\frac{pc_1}{c(a_1d_1-b_1c_1)} \tag{14d}$$

with a_1, b_1, c_1, and d_1 being given by Eqs. (A5) and

$$B_2=B_3=0$$

because w is a symmetric function of x.

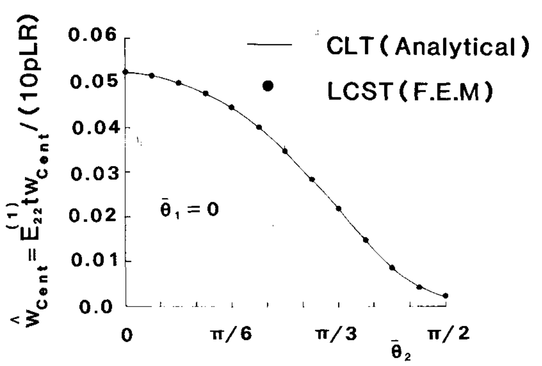

Fig. 2 Variation of central transverse displacement of a two-layer cylindrical shell with respect to the fiber orientation of the outer layer.

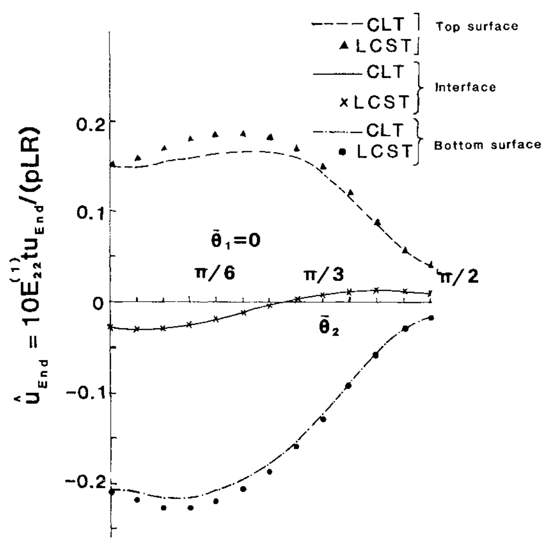

Fig. 3 Variation of end longitudinal displacement of a two-layer cylindrical shell with respect to the fiber orientation of the outer layer.

Once the three displacement components are known, the strains and curvature changes and, finally, the surface parallel stresses can be determined. Transverse stresses can be obtained by using Eqs. (2.11.1) of Seide.[16]

Numerical Results

The results on laminated anisotropic shells so far available in the literature are restricted to such symmetries of lamination as unidirectional, cross ply, and angle ply. The present study will investigate, as an example, a two-layer asymmetrically laminated cylindrical shell involving no such symmetry of lamination. Simply supported condition SS1 is considered here. The inner layer of the shell has fibers oriented in the direction parallel to the axis of the cylinder (x axis). The fiber orientation in the outer layer is varied at 0–90 deg with respect to the x axis. The length of the cylindrical shell and the inner radius are 508 and 254 mm (20 and 10 in.), respectively, while the thickness of each layer is 2.54 mm (0.1 in.). E_{11} and E_{22}, Young's moduli in the directions parallel and transverse to the fibers, are 275.8 and 6.895 GPa (40×10^6 and 10^6 psi), respectively. G_{12} and $G_{13}=G_{23}$ (in-plane and transverse shear moduli, respectively) are both assumed to be equal to 3.448 GPa (0.5×10^6 psi). Major Poisson's ratio, $\nu_{12}=\nu_{13}$, defined to be ratio of the strains perpendicular and parallel to the fiber direction caused by the stress in the direction of fibers, is taken equal to 0.25. Transverse major Poisson's ratio ν_{23} is also assumed to be the same.

Figures 2–7 show variation of transverse displacement and surface-parallel stresses at the central section of the cylinder (i.e., $x=0$) as well as variation of surface-parallel displacements at the end (i.e., $x=L/2$), with respect to the fiber orientation of the outer layer. The present CLT solutions are compared with those obtained by the finite element method (FEM), based on the assumptions of transverse inextensibility and LCST. Details on formulation and convergence of the element are available in Refs. 4 and 14 and are not repeated here. These figures suggest close agreement between the two solutions for a thin shell at a point away from the edge. It is further noteworthy that the circumferential displacement component obtained by CLT does not vary through the thickness, while the variation obtained by LCST

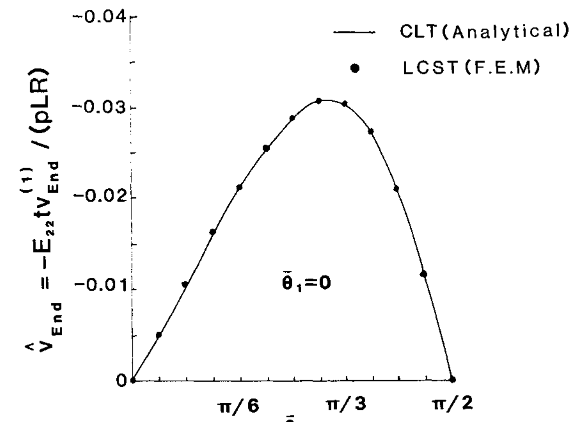

Fig. 4 Variation of end circumferential displacement of a two-layer cylindrical shell with respect to the fiber orientation of the outer layer.

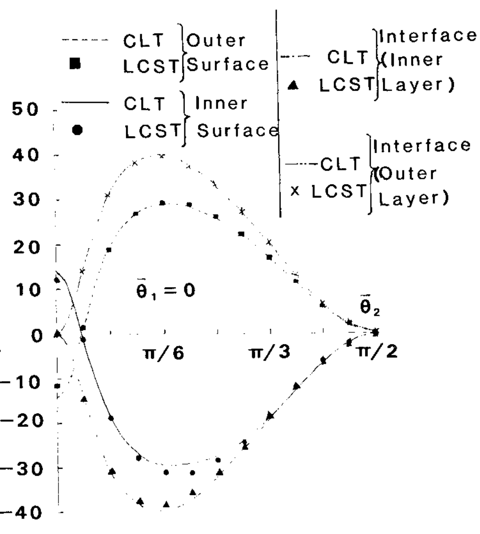

Fig. 5 Variation of central longitudinal stress of a two-layer cylindrical shell with respect to the fiber orientation of the outer layer.

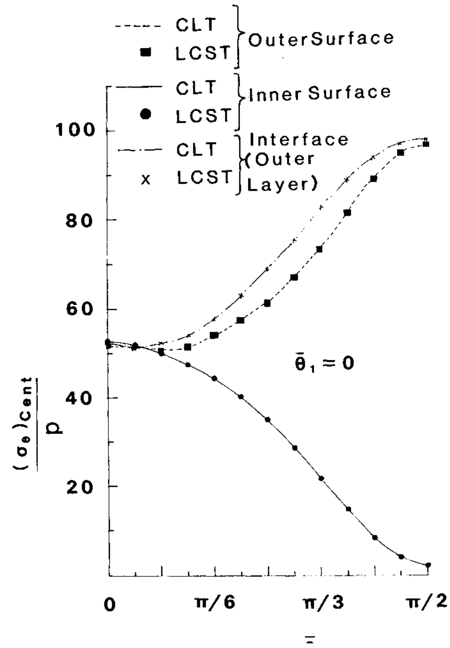

Fig. 6 Variation of central circumferential stress of a two-layer cylindrical shell with respect to the fiber orientation of the outer layer.

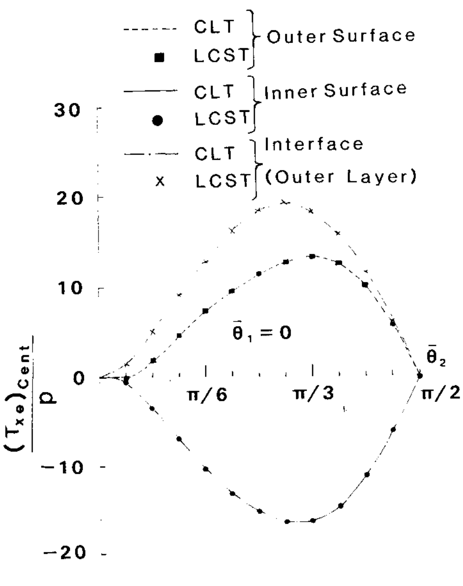

Fig. 7 Variation of central surface-parallel shear stress of a two-layer cylindrical shell with respect to the fiber orientation of the outer layer.

can be considered to be almost constant for the cylindrical shell under investigation. It is further interesting to note that σ_θ and $\tau_{x\theta}$ at the top and bottom surfaces of the inner layer are almost identical, while the same is not true for σ_x at the top and bottom surfaces of the inner layer. The reason behind this seemingly paradoxical behavior is that the stresses σ_θ and $\tau_{x\theta}$ for the inner layer are primarily membrane stresses, while stresses σ_x for the same layer are due to both bending and membrane, with bending being predominant for $\bar{\theta}_2 < 60$ deg approximately. For $\bar{\theta} \geq 60$ deg (approximately), σ_x for the inner layer are membrane stresses. In general, the stresses in the outer layer seem to behave differently than those of the inner layer. Exceptions to this include σ_x for the outer layer behaving like membrane stresses for $\bar{\theta}_2 \leq 75$ deg (approximately), σ_θ becoming almost membrane-like for $\bar{\theta}_2 \sim 0$ and 90 deg, and $\tau_{x\theta}$ for the outer layer, understandably, vanishing for $\bar{\theta}_2 = 0$ and 90 deg. The difference in the behavior of the stresses in the inner and outer layers is, to a small extent, due to the effect of curvature, but primarily because the outer layer has fibers oriented at an angle, which produces such coupling effect as, for example, bending/twisting (except when $\bar{\theta}_2 = 0$ or 90 deg). For example, the stresses $\tau_{x\theta}$, for the outer layer with $\bar{\theta}_2 = 45$ deg, are due to both in-plane shear and twisting in contrast to the same for the inner layer, which are primarily due to membrane shear.

Figures 8–13 show the variation of displacements w, u_0, and v_0 and stresses σ_x, σ_θ, and $\tau_{x\theta}$ along the axial direction of the cylindrical shell with $\bar{\theta}_2 = 45$ deg. These plots indicate that, while the effect of transverse shear deformation on radial displacement of the thin asymmetrically laminated shell is not significant, the maximum stresses computed using LCST are about 10% higher than those due to CLT. The disagreement is, nonetheless, confined to the region of the

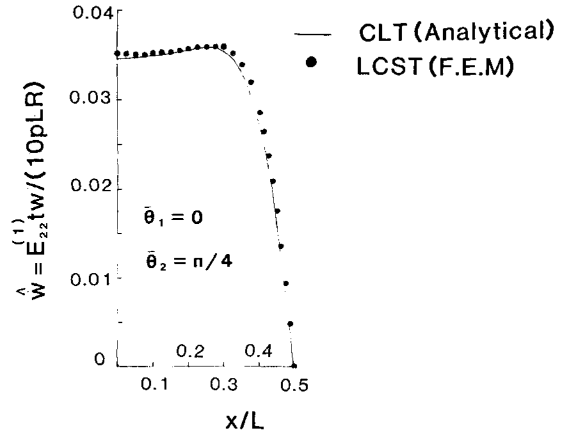

Fig. 8 Axial variation of transverse displacement of a two-layer (0/45 deg) cylindrical shell.

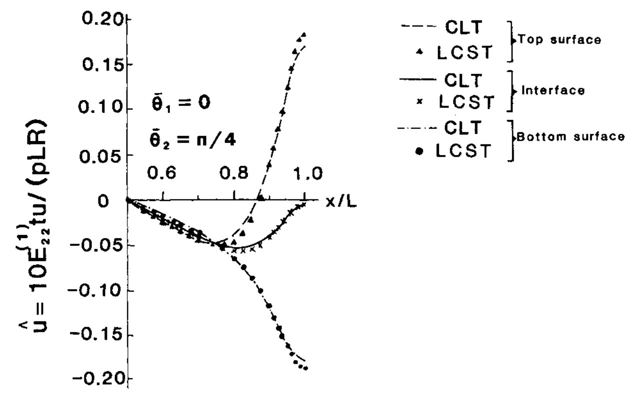

Fig. 9 Axial variation of longitudinal displacement of a two-layer (0/45 deg) cylindrical shell.

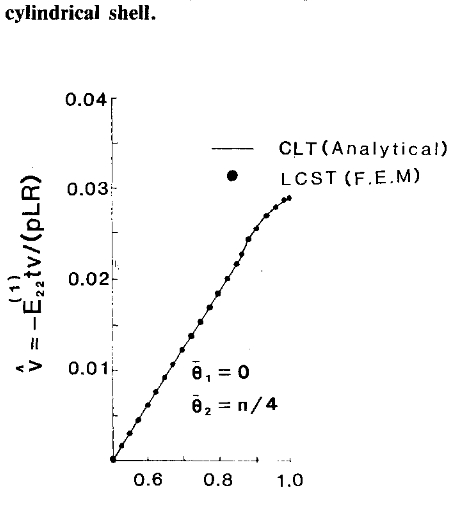

Fig. 10 Axial variation of circumferential displacement of a two-layer (0/45 deg) cylindrical shell.

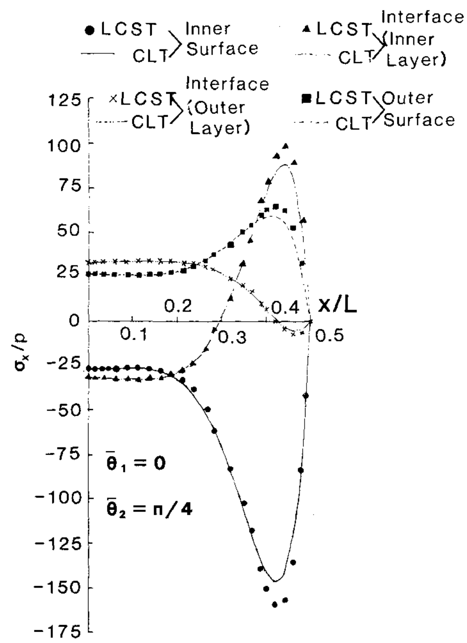

Fig. 11 Axial variation of longitudinal stress of a two-layer (0/45 deg) cylindrical shell.

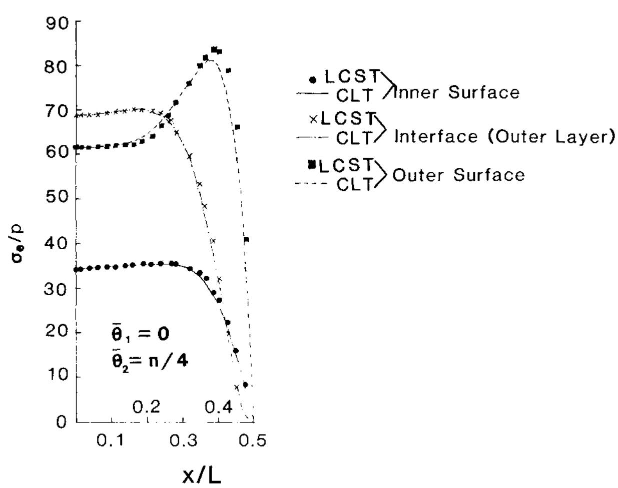

Fig. 12 Axial variation of circumferential stress of a two-layer (0/45 deg) cylindrical shell.

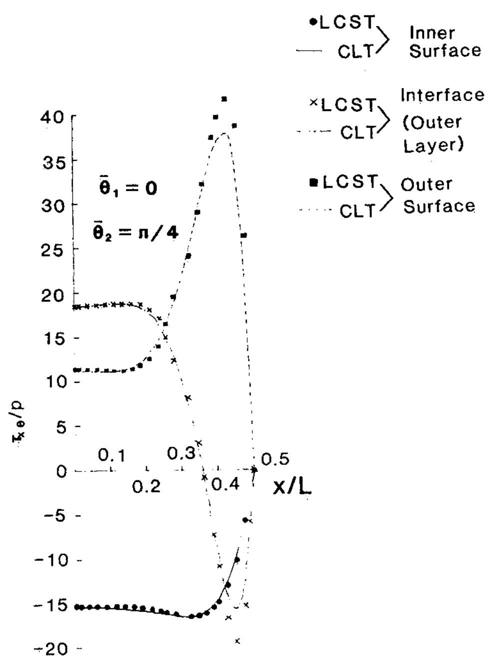

Fig. 13 Axial variation of surface-parallel shear stress of a two-layer (0/45 deg) cylindrical shell.

shell characterized by the boundary-layer effect. Furthermore, the disagreement between the two solutions is more pronounced for the curves with sharper peaks and for the cases of stresses at the top and bottom surface of the layer having opposite signs, such that the moments or stress couples (not shown), computed using these stresses, are in close agreement.

Conclusions

A closed-form solution for an asymmetrically laminated anisotropic cylindrical shell of finite length under uniform internal pressure is presented, using Love-Timoshenko's kinematic relations and CLT (based on Love's first approximation theory). The numerical results for a two-layer shell demonstrate the influence of the various coupling effects induced by the asymmetry of lamination and anisotropy of individual laminae. Comparison of these CLT-based results with the LCST-based finite element solutions suggest that, while there is close agreement between the two solutions in computing displacements and moments, slight disagreement (approximately 10%) has been observed in computing the maximum stresses in individual layers, which is confined to the "boundary-layer" zone of the shell. Solutions presented herein should serve as benchmark results for future comparisons and facilitate the use of asymmetrically laminated shell in design.

Appendix A

The constants a, b, c, and p' referred to in Eq. (6) are written as follows:

$$a = D_{11} - \left\{\left[B_{11}^2\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - 2B_{11}\left(B_{16}+\frac{2D_{16}}{R}\right)\left(A_{16}+\frac{2B_{16}}{R}\right) + A_{11}\left(B_{16}+\frac{2D_{16}}{R}\right)^2\right] \div \left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - \left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]\right\} \quad \text{(A1a)}$$

$$b = -\frac{2B_{12}}{R} + \left\{2\left[A_{12}B_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) + A_{11}\left(B_{16}+\frac{2D_{16}}{R}\right)\left(A_{26}+\frac{2B_{26}}{R}\right)\right] \div R\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - \left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]\right\}$$
$$- \left\{2\left[A_{12}\left(B_{16}+\frac{2D_{16}}{R}\right)\left(A_{16}+\frac{2B_{16}}{R}\right) - B_{11}\left(A_{16}+\frac{2B_{16}}{R}\right)\left(A_{26}+\frac{2B_{26}}{R}\right)\right] \div R\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - \left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]\right\} \quad \text{(A1b)}$$

$$c = \frac{A_{22}}{R^2} - \left\{\left[A_{12}^2\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - 2A_{12}\left(A_{16}+\frac{2B_{16}}{R}\right)\left(A_{26}+\frac{2B_{26}}{R}\right) + A_{11}\left(A_{26}+\frac{2B_{26}}{R}\right)^2\right] \div R^2\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - \left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]\right\} \quad \text{(A1c)}$$

$$p' = p - \frac{A_{12}}{R}\left\{\left[A_1\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - A_2\left(A_{16}+\frac{2B_{16}}{R}\right)\right] \div \left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - \left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]\right\}$$
$$+ \frac{1}{R}\left(A_{26}+\frac{2B_{26}}{R}\right)\left\{\left[A_2A_{11} - A_1\left(A_{16}+\frac{2B_{16}}{R}\right)\right] \div \left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - \left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]\right\} \quad \text{(A1d)}$$

For orthotropic layers, these reduce to[10]

$$a = D_{11} - (B_{11}^2/A_{11}) \quad \text{(A2a)}$$

$$b = (2/\overline{RA}_{11})(A_{12}B_{11} - B_{12}A_{11}) \quad \text{(A2b)}$$

$$c = (A_{11}A_{22} - A_{12}^2)/\overline{R^2A}_{11} \quad \text{(A2c)}$$

$$p' = p - (A_1A_{12}/\overline{RA}_{11}) \quad \text{(A2d)}$$

The constants a', b', c', d' and e', f', g', h' referred to in Eqs. (9) and (10), respectively, are

$$a' = \frac{\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)}{\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - \left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]} \quad \text{(A3a)}$$

$$b' = \frac{\left(A_{16}+\frac{2B_{16}}{R}\right)}{\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right) - \left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]} \quad \text{(A3b)}$$

$$c' = -\frac{\left[A_{12}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)-\left(A_{16}+\frac{2B_{16}}{R}\right)\left(A_{26}+\frac{2B_{26}}{R}\right)\right]}{(\alpha^2+\beta^2)R\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)-\left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]} \tag{A3c}$$

$$d' = \frac{\left[B_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)-\left(A_{16}+\frac{2B_{16}}{R}\right)\left(B_{16}+\frac{2D_{16}}{R}\right)\right]}{\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)-\left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]} \tag{A3d}$$

$$e' = \frac{\left(A_{16}+\frac{2B_{16}}{R}\right)}{\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)-\left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]} \tag{A4a}$$

$$f' = \frac{A_{11}}{\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)-\left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]} \tag{A4b}$$

$$g' = \frac{\left[-A_{11}\left(A_{26}+\frac{2B_{26}}{R}\right)+A_{12}\left(A_{16}+\frac{2B_{16}}{R}\right)\right]}{R(\alpha^2+\beta^2)\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)-\left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]} \tag{A4c}$$

$$h' = \frac{\left[A_{11}\left(B_{16}+\frac{2D_{16}}{R}\right)-B_{11}\left(A_{16}+\frac{2B_{16}}{R}\right)\right]}{R\left[A_{11}\left(A_{66}+\frac{4B_{66}}{R}+\frac{4D_{66}}{R^2}\right)-\left(A_{16}+\frac{2B_{16}}{R}\right)^2\right]} \tag{A4d}$$

The constant terms a_1, b_1, c_1, and d_1 referred to in Eqs. (14) can be written as

$$a_1 = \sin\left(\frac{\alpha L}{2}\right)\sinh\left(\frac{\beta L}{2}\right) \tag{A5a}$$

$$b_1 = \cos\left(\frac{\alpha L}{2}\right)\cosh\left(\frac{\beta L}{2}\right) \tag{A5b}$$

$$c_1 = (\beta^2-\alpha^2)\sin\left(\frac{\alpha L}{2}\right)\sinh\left(\frac{\alpha L}{2}\right) + 2\alpha\beta\cos\left(\frac{\alpha L}{2}\right)\cosh\left(\frac{\beta L}{2}\right) \tag{A5c}$$

$$d_1 = (\beta^2-\alpha^2)\cos\left(\frac{\alpha L}{2}\right)\cosh\left(\frac{\beta L}{2}\right) - 2\alpha\beta\sin\left(\frac{\alpha L}{2}\right)\sinh\left(\frac{\beta L}{2}\right) \tag{A5d}$$

Appendix B: Justification for Use of the Terminology of "Love-Timoshenko's Kinematic Relations"

Classical lamination theory (CLT) for laminated composite shells is an extension of Love's first approximation theory[13] or Love-Kirchhoff theory, because this is based on Love's postulates. However, within the framework of CLT, a number of kinematic relations relating the strains and curvatures on the reference surface to the reference surface displacements (for example, those due to Donnell, Sanders, and Love-Timoshenko) are available. Love[13] has presented these relations for a general shell from which Timoshenko and Woinowsky-Krieger[15] derived the kinematic relations for a cylindrical shell. Kraus[19] presents Reissner's version of Love's first approximation theory, which is different from Love's theory as derived by Love himself (see Ref. 19, footnote, p. 24). The twist term χ_{12} as given by Kraus[19] [Eq. (6.8c), p. 200] is

$$(\chi_{12})_{\substack{\text{present}\\\text{paper}}} = (\tau)_{\text{Kraus}} = (1/R)(-2w,_{x\theta}+v_{0,x}) \tag{B1}$$

while the same term as presented by Timoshenko and Woinowsky-Krieger[15] [see p. 432 and Eq. (300), p. 512] is

$$(\chi_{12})_{\substack{\text{present}\\\text{paper}}} = 2(\chi_{x\phi})_{\text{Timo}} = (2/R)(-w,_{x\theta}+v_{0,x}) \tag{B2}$$

Since the present paper is concerned with a cylindrical shell and uses Eq. (B2), it is considered highly appropriate, in order to avoid confusion, to mention Timoshenko's name in addition to that of Love and also give reference to both the books. Admittedly, Timoshenko has presented a beam theory that includes transverse shear deformation, but his work on shell theory excludes the shear deformation effect—a view also shared by the reviewer. In our opinion, since the CLT based on Love's first approximation theory or Love-Kirchhoff theory excludes the shear deformation effect, anyone familiar with shell theory would not confuse Love-Timoshenko's kinematic relations with the shear deformation theory. In this regard, even if there is a popular misconception, as the reviewer believes, we feel that it is important to set the record straight and not to perpetuate this misconception.

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